

① Show $\mathcal{U} = \{ f \in C[0,1] \mid f(x) > 0 \ \forall x \in [0,1] \}$ is open

② Let $S = \{ f \in C_b(\mathbb{R}) \mid f(x) > 0 \ \forall x \in \mathbb{R} \}$.

What is $\text{int } S$?

③ Show that $\mathcal{F} = \left\{ F(x) = \int_0^x f(t) dt \mid f \in C[0,1], \|f\|_\infty \leq 1 \right\}$

is uniformly bounded and equicontinuous, but not closed.

Show $\overline{\mathcal{F}} = \{ g \in C[0,1] \mid g(0) = 0, |g(x) - g(y)| \leq |x - y| \}$

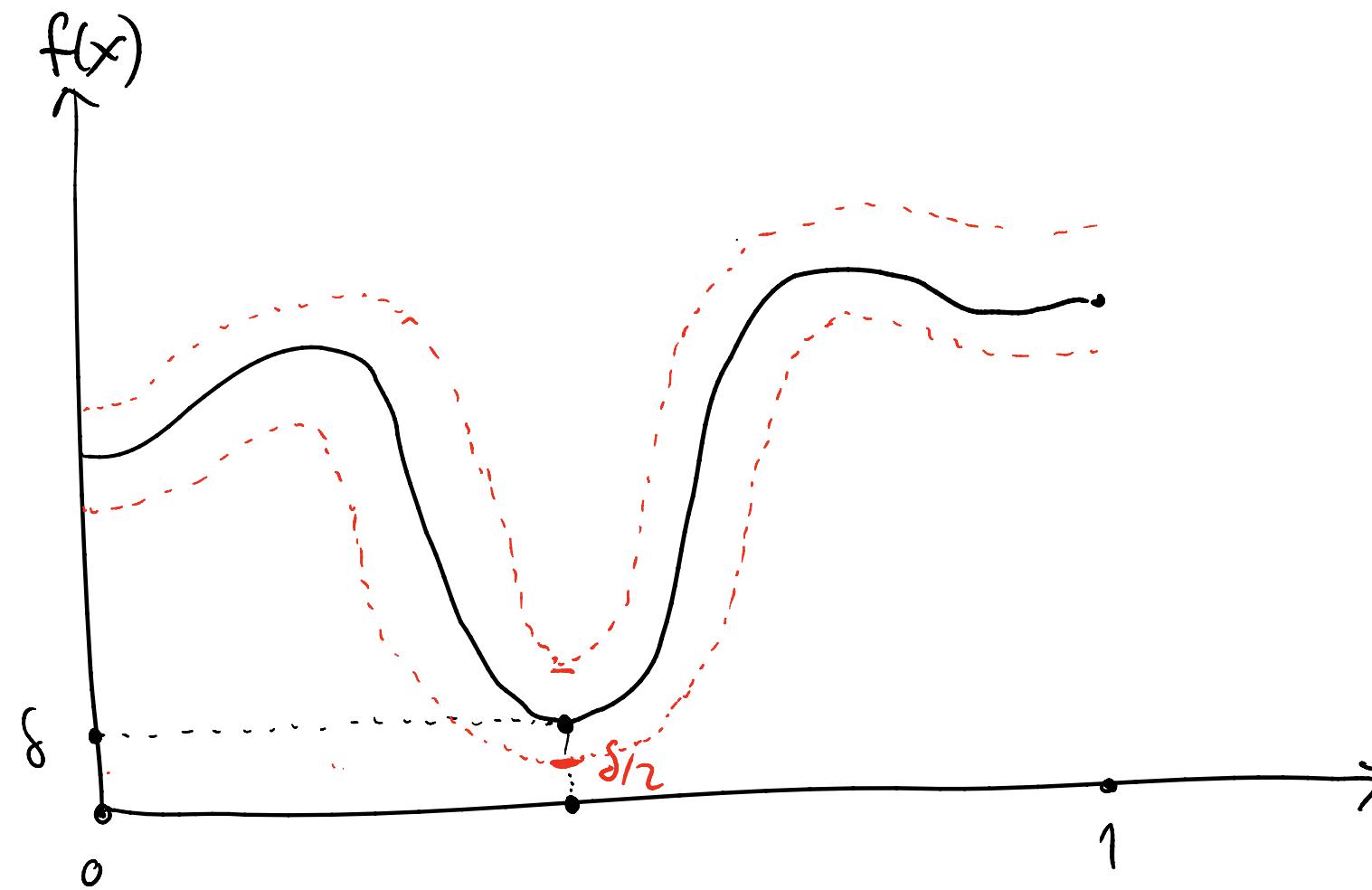
④ Are the following families equicontinuous?

$\mathcal{F}_1 = \left\{ f \in C[0,1] \mid f(x) = \sum_{j \leq 100} a_j x^j, a_j \in \mathbb{R} \right\}$

$\mathcal{F}_2 = \left\{ f \in C[0,2\pi] \mid f(x) = \sum_{j=-N}^N a_j \cos(jx) + b_j \sin(jx), \forall j \quad |a_j| \leq M, |b_j| \leq M \right\}$

st.

Problem 1 We will proceed using the definition^{v of an open set.} Choose $f \in U$, so $f(x) > 0 \quad \forall x \in [0, 1]$. Since f is continuous and $[0, 1]$ is compact, $f([0, 1]) \subseteq \{y > 0\} \subseteq \mathbb{R}$ is compact. $\exists \delta \in f([0, 1])$ s.t. $f(x) \geq \delta > 0 \quad \forall x \in [0, 1]$.



Now consider

$$B(f, \delta/2) = \left\{ g \in C[0,1] \mid \sup_{x \in [0,1]} |f(x) - g(x)| < \delta/2 \right\}$$

These are functions g whose graphs stay inside the red band.

If we show $B(f, \delta/2) \subseteq U$ we are done; i.e. need to

show $\forall g \in B(f, \delta/2)$ we have $g(x) > 0$.

$$\begin{aligned} g(x) &= f(x) - (f(x) - g(x)) && \left(f(x) - g(x) \leq \delta/2 \Rightarrow -(f(x) - g(x)) > -\delta/2 \right) \\ &> f(x) - \delta/2 \\ &\geq \delta - \delta/2 \\ &= \delta/2 \\ &> 0 \end{aligned}$$

Therefore $g(x) > 0 \quad \forall x \in [0,1]$.

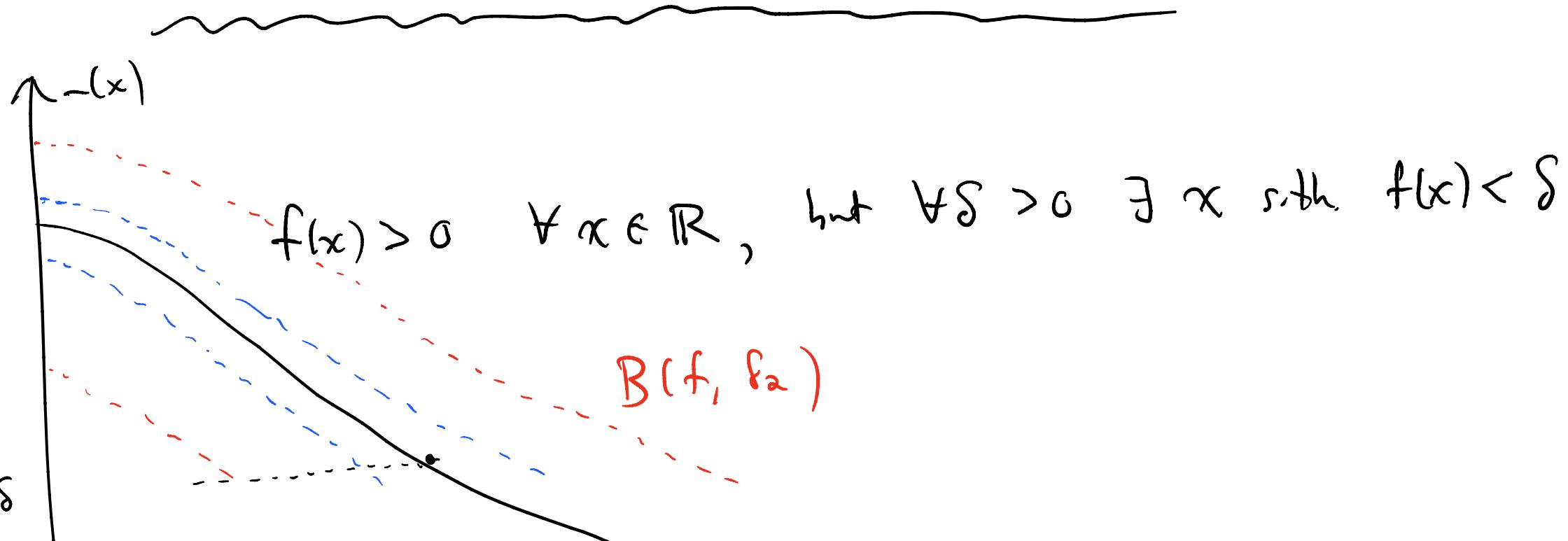
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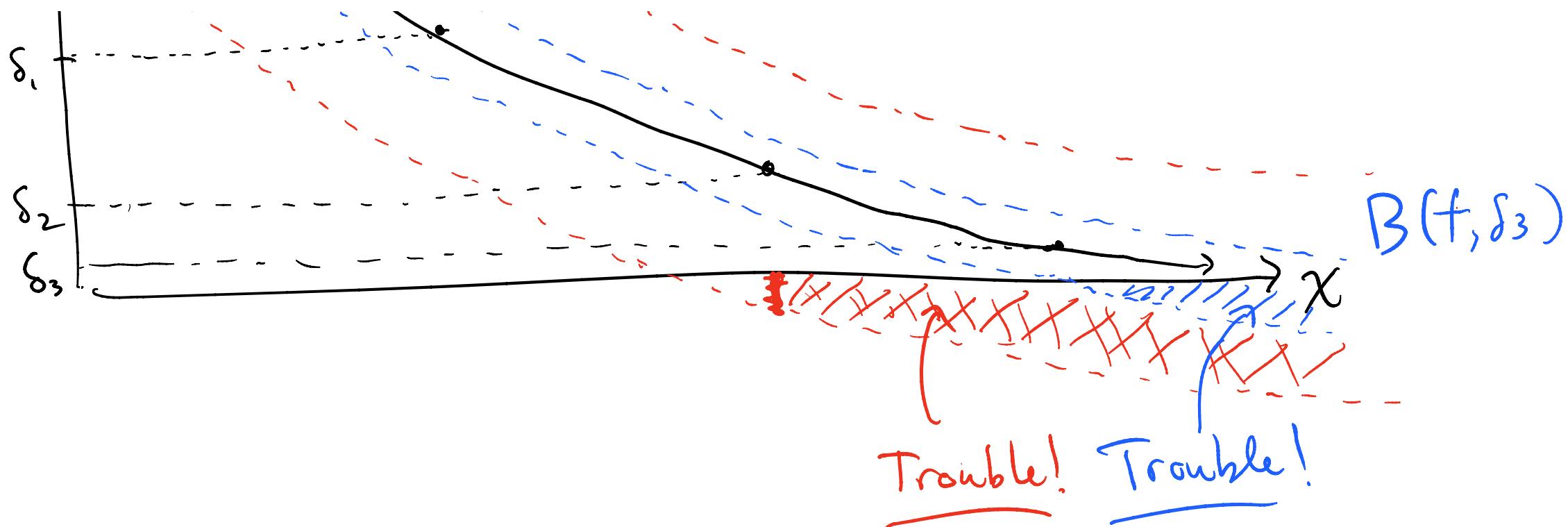
Problem 2

This is almost the same as Q1; the crucial difference is that we are looking at continuous functions on \mathbb{R} , and not a compact subset thereof.

What can go wrong with S being open? Our proof before relied on the ability to bound f away from zero uniformly by an amount δ .

We cannot do this if $\lim_{x \rightarrow \pm\infty} f(x) = 0$





Prop: $S^{\text{int}} = \bigcup_{\delta > 0} \{g \in C_b(\mathbb{R}) \mid g(x) \geq \delta \forall x \in \mathbb{R}\}$

Pf: (2) If $f \in \bigcup_{\delta > 0} \{g \in C_b(\mathbb{R}) \mid g(x) \geq \delta \forall x \in \mathbb{R}\}$ then $\exists \delta > 0$

s.t. $\forall x \in \mathbb{R} \quad f(x) \geq \delta$. The proof now proceeds as in

problem 1 - i.e. $B(f, \delta/2) \subseteq S$

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(\Leftarrow) If $f \in S'''$ then $\exists \delta > 0$ s.t. $D(f, \delta) = S$

In particular $\forall g \in B(f, \delta)$, we have $g(x) > 0$

Clearly, $f(x) - \delta \in B(f, \delta)$ so

$$f(x) - \delta > 0 \Rightarrow f(x) > \delta \geq \delta/2$$

So $f \in \bigcup_{\delta > 0} \{g \in C_b(\mathbb{R}) \mid g(x) \geq \delta \quad \forall x \in \mathbb{R}\}$

Problem 3 $\mathcal{F} = \left\{ F(x) = \int_0^x f(t) dt \mid f \in C[0, 1], \|f\|_\infty \leq 1 \right\} \subseteq C[0, 1]$.

Uniformly bounded: We claim $\|F\|_\infty \leq 1 \quad \forall F \in \mathcal{F}$

$$\|F\|_\infty = \sup_{x \in [0, 1]} \left| \int_0^x f(t) dt \right|$$

$$\int_0^x 1$$

$$\begin{aligned}
 &\leq \sup_{x \in [0,1]} \int_0^x \|f(t)\| dt \\
 &\leq \sup_{x \in [0,1]} \int_0^x dt \quad \text{since } \|f\|_\infty \leq 1 \\
 &= \sup_{x \in [0,1]} x \\
 &= 1
 \end{aligned}$$

$$S_0 \|F\|_\infty \leq 1 \quad \forall F \in \mathcal{F}$$

Equicontinuity: \mathcal{F} is equicontinuous $\Leftrightarrow \forall x \in S \ \forall \varepsilon > 0, \exists \delta > 0, \forall F \in \mathcal{F} \ \forall y \in S$
 on S $|x-y| < \delta \Rightarrow |F(x) - F(y)| < \varepsilon$

Lemma If $F \in \mathcal{F}$ then $\forall x, y \in [0,1] \ |F(x) - F(y)| \leq |x-y|$.

Choose an $F \in \mathcal{F}$, any two $x, y \in [0,1]$ and estimate.

$$|F(x) - F(y)| = \left| \int_0^x f(t) dt - \int_0^y f(t) dt \right| \quad \text{suppose } x > y$$

$$\begin{aligned}
 &= \left| \int_y^x f(t) dt \right| \\
 &\leq \int_y^x |f(t)| dt \\
 &\leq \int_y^x 1 dt
 \end{aligned}$$

$$= x - y$$

if $y > x$ the inequality works similarly. In either case,

$\forall F \in \mathcal{F}$ and $x, y \in [a, b]$.

$$|F(x) - F(y)| \leq |x - y|$$

□

Now we show \mathcal{F} is equicontinuous.

Fix $x \in [0,1]$ and any $\varepsilon > 0$. If $F \in \mathcal{F}$ and $y \in [0,1]$, then

$$\begin{aligned} |F(x) - F(y)| &\leq |x - y| \\ &< \delta \quad \text{choose } \delta = \varepsilon \\ &= \varepsilon \end{aligned}$$

So \mathcal{F} is equicontinuous.

Remark: Any family satisfying a uniform Lipschitz estimate is equicontinuous. This is often very useful.

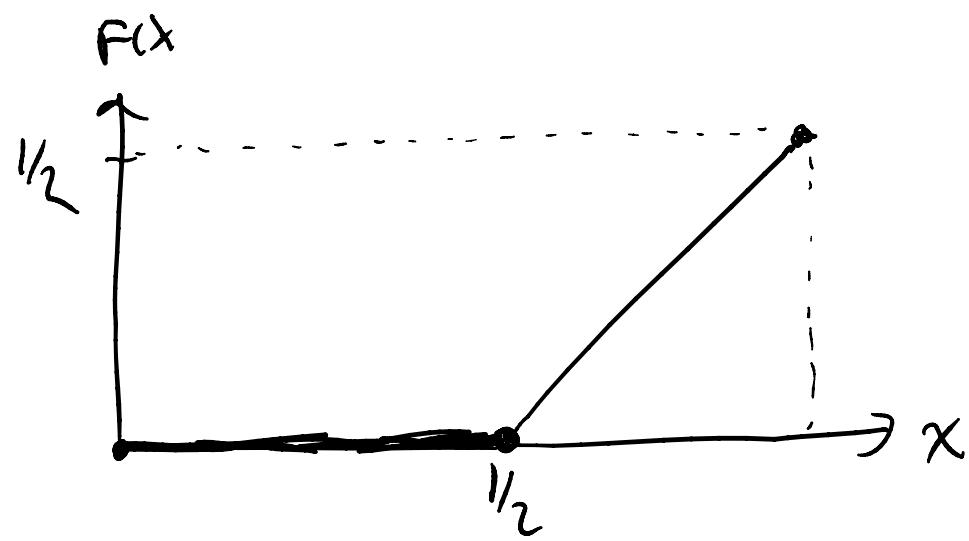
let's show \mathcal{F} is not closed

Notice that the FTC implies that any $F \in \mathcal{F}$ is differentiable.

It suffices to find a non differentiable function and express it as a uniform limit of a sequence $\{F_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$.

as a uniform limit of a sequence $\{t_n\}_{n=1} \subseteq J$.

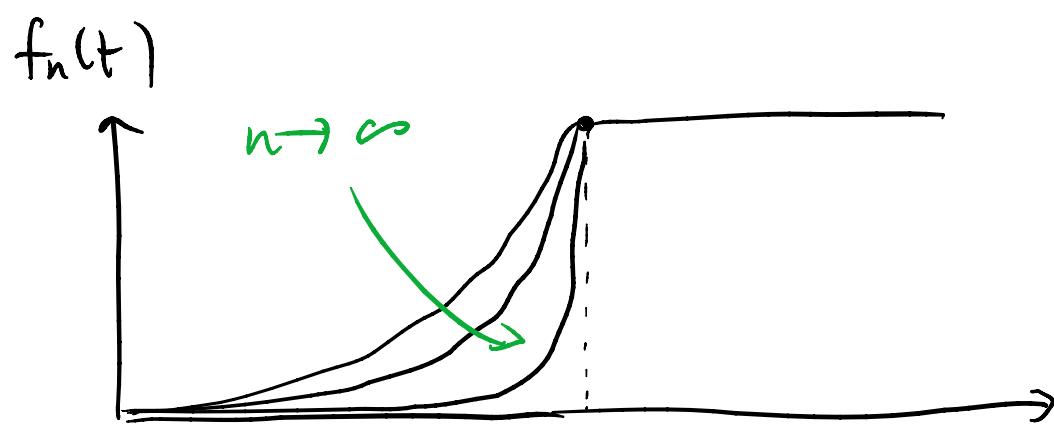
e.g. Consider $F(x)$ given below



$$F(x) = \int_0^x f(t) dt$$

where $f(t) = \begin{cases} 0 & 0 \leq t < 1/2 \\ 1 & 1/2 \leq t \leq 1 \end{cases}$

$$\text{let } f_n(t) = \begin{cases} (2t)^n & 0 \leq t < 1/2 \\ 1 & 1/2 \leq t \leq 1 \end{cases}$$



Notice $f_n(t) \rightarrow f(t)$ pointwise.

$$\int_0^x$$

$$\text{let } F_n(x) = \int_0^x f_n(t) dt$$

$$= \begin{cases} \frac{(2x)^{n+1}}{2(n+1)} & 0 \leq x < 1/2 \\ \frac{1}{2(n+1)} + x - 1/2 & 1/2 \leq x \leq 1 \end{cases}$$

Prop: $F_n \rightarrow F$ uniformly

Pf: Fix $\epsilon > 0$ and estimate.

If $0 \leq x < 1/2$

$$|F_n(x) - F(x)| = \left| \frac{(2x)^{n+1}}{2(n+1)} \right| < \frac{1}{2(n+1)} \quad x < 1/2$$

$< \epsilon$ choose n large.

If $1/2 \leq x \leq 1$

$$|F_n(x) - F(x)| = \left| \left(\frac{1}{2(n+1)} + x - 1/2 \right) - (x - 1/2) \right| = \frac{1}{2(n+1)} < \epsilon$$

$\rightarrow c \cdots 0$

So $f_n \rightarrow f$ uniformly. \square

Now we have shown:

① $F_n \in \mathcal{F} \forall n$

② $F \notin \mathcal{F}$ since F is not differentiable

③ $f_n \rightarrow f$ uniformly

① + ② + ③ implies \mathcal{F} is not closed.